Reassembling Trees for the Traveling Salesman

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TSP variants – state of the art

Integrality ratios. Upper bounds = approximation ratios unless mentioned otherwise

2ECSS, general weights:

- between $\frac{6}{5}$ and $\frac{3}{2}$ (Alexander, Boyd, Elliott-Magwood [2006])

2ECSS, unit weights:

- between $\frac{8}{7}$ (Boyd, Fu, Sun [2014]) and $\frac{4}{3}$ (Sebő, V. [2012])

TSP, general weights:

- between $\frac{4}{3}$ and $\frac{3}{2}$ (Wolsey [1980])

TSP, unit weights:

- between $\frac{4}{3}$ and $\frac{7}{5}$ (Sebő, V. [2012])

s-t-path TSP, general weights:

- between $\frac{3}{2}$ and $\frac{8}{5}$ (Sebő [2013])

s-t-path TSP, unit weights:

- $\frac{3}{2}$ (Sebő, V. [2012])

ATSP, general weights:

- between 2 (Boyd, Elliott-Magwood [2005], Charikar, Goemans, Karloff [2006]) and $\log^{O(1)} \log n$ (Anari and Oveis Gharan [2014]); apx ratio $8 \log n / \log \log n$ (Asadpour, Goemans, Mądry, Oveis Gharan, Saberi [2010])

ATSP, unit weights:

- between $\frac{3}{2}$ (Gottschalk [2013]) and 13; apx ratio $27 + \epsilon$ (Svensson [2015])

s-t-path TSP

"Start at s, visit all cities, end at t, minimize total distance."

Instance:

- a finite set V (of cities),
- two cities $s, t \in V$ ($s \neq t$), and
- a metric $c : V \times V \to \mathbb{R}_{\geq 0}$

Task: find

- a sequence $V = \{v_1, \dots, v_n\}$ with $s = v_1$ and $t = v_n$
- such that $\sum_{i=1}^{n-1} c(v_i, v_{i+1})$ is minimized.

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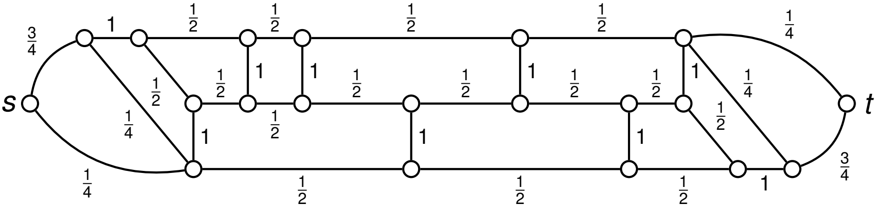
Previous approximation algorithms:

- 2 (double-tree algorithm) (folklore)
- $\frac{5}{3}$ (Christofides' algorithm) (Hoogeveen [1991])
- $\frac{1+\sqrt{5}}{2} \approx 1.619$ (best-of-many Christofides) (An, Kleinberg, Shmoys [2012])
- $\frac{8}{5}$ (best-of-many Christofides) (Sebő [2013])

LP relaxation

$$E := \binom{V}{2}, \quad c(x) := \sum_{e=\{v,w\}\in E} c(v,w)x_e, \quad x(F) := \sum_{e\in F} x_e.$$

$$\begin{array}{llcll}
\min\ c(x) & & & & \\
\text{subject to} & x(\delta(U)) & \geq & 2 & (\emptyset \neq U \subset V,\ |U \cap \{s,t\}| \text{ even}) \\
 & x(\delta(U)) & \geq & 1 & (\emptyset \neq U \subset V,\ |U \cap \{s,t\}| \text{ odd}) \\
 & x(\delta(v)) & = & 2 & (v \in V \setminus \{s,t\}) \\
 & x(\delta(v)) & = & 1 & (v \in \{s,t\}) \\
 & x_e & \geq & 0 & (e \in E)
\end{array}$$



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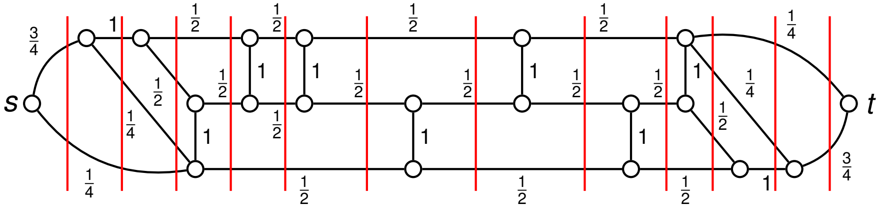


$\mathcal{C} := \{C = \delta(U) : x(C) < 2\}$ (narrow cuts, form a chain)

Best-of-Many-Christofides (An, Kleinberg, Shmoys [2012])

- Solve the LP, let x^* be an optimum solution.
- Decompose x^* into spanning trees: write

$$x^* = \sum_{S \in \mathcal{S}} p_S \chi^S$$

  where $\mathcal{S}$ is the set of edge sets of spanning trees, $p_S \geq 0$ ($S \in \mathcal{S}$) and $\sum_{S \in \mathcal{S}} p_S = 1$.
  (Edmonds [1970], Held, Karp [1970], Grötschel, Lovász, Schrijver [1981], Frank [2011], Genova, Williamson [2015])
- Do parity correction for each $S \in \mathcal{S}$ with $p_S > 0$:
  add a minimum cost T_S-join,
  where T_S contains the vertices whose degree in S has the wrong parity (even for s or t, odd for other vertices).
  (Edmonds [1965], Christofides [1976])
- Take the best of these tours. Shortcut if cities are visited more than once.

Basic Analysis (An, Kleinberg, Shmoys [2012])

The result has cost

$$\begin{aligned}
& \min_{S \in \mathcal{S}:\, p_S > 0} \big(c(S) + \min\{c(J) : J \text{ is a } T_S\text{-join}\} \big) \\
\leq \;& \sum_{S \in \mathcal{S}} p_S \big(c(S) + \min\{c(J) : J \text{ is a } T_S\text{-join}\} \big) \\
= \;& c(x^*) + \sum_{S \in \mathcal{S}} p_S \min\{c(J) : J \text{ is a } T_S\text{-join}\} \\
\leq \;& c(x^*) + \sum_{S \in \mathcal{S}} p_S c(y^S)
\end{aligned}$$

for any set of correction vectors y^S ($S \in \mathcal{S}$) such that y^S is in the T_S-join polyhedron

$$\left\{ y \in \mathbb{R}^E_{\geq 0} : y(C) \geq 1 \;\forall\; T_S\text{-cuts } C \right\}.$$

(Edmonds, Johnson [1973])

Example: x^* is a correction vector for every S.

Correction vectors (An, Kleinberg, Shmoys [2012], Sebő [2013])

Let $S = I_S \dot\cup J_S$, where I_S is the s-t-path and J_S is the T_S-join. Let

$$y^S \;:=\; (1-2\beta)\chi^{J_S} + \beta x^* + r^S$$

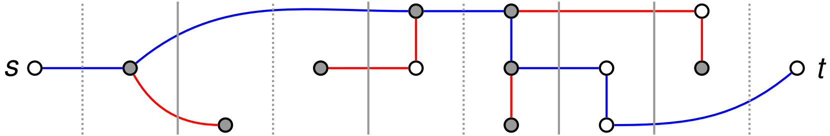
for $S \in \mathcal{S}$, where $0 \le \beta \le \frac{1}{2}$, and $r^S \in \mathbb{R}^E_{\ge 0}$ satisfies

$$r^S(C) \;\ge\; \beta(2 - x^*(C))$$

for all $S \in \mathcal{S}$ and all (narrow) cuts C with $|S \cap C|$ even.

Then, for every $S \in \mathcal{S}$ and every T_S-cut C we have

$$y^S(C) \;\ge\; 1.$$



$S = I_S \dot\cup J_S$. Narrow cuts (grey) that need parity correction (solid) contain (at least) one red and one blue edge.

András's correction vectors (Sebő [2013])

Again, $S = I_S \,\dot{\cup}\, J_S$, where I_S is the s-t-path and J_S is the T_S-join.
As above,

$$y^S \;:=\; (1-2\beta)\chi^{J_S} + \beta x^* + r^S$$

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$$r^S(C) \;\geq\; \beta(2 - x^*(C))$$

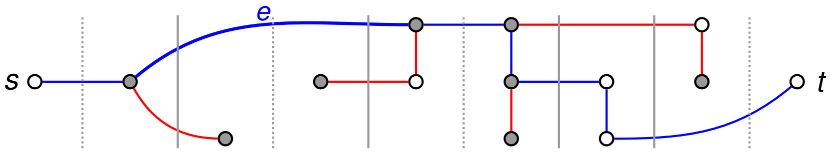
for all $S \in \mathcal{S}$ and all $C \in \mathcal{C}$ with $|S \cap C|$ even, and

$$\sum_{S\in\mathcal{S}} p_S r^S \;\leq\; (1-2\beta)\sum_{S\in\mathcal{S}} p_S \chi^{I_S}.$$

Implies approximation ratio $1-\beta$. Sebő [2013] obtained $\beta = \frac{2}{5}$.

New correction vectors

For every $S \in \mathcal{S}$ and every $e \in I_S$, we can distribute $(1-2\beta)p_S$ to the correction vectors.



Sebő [2013]:

Half goes to y^S, repairing cuts C with $e \in C$ and $|S \cap C|$ even.

Half goes into a box for the cut C with $S \cap C = \{e\}$.

If y^S needs more to repair even cut C, take from box.

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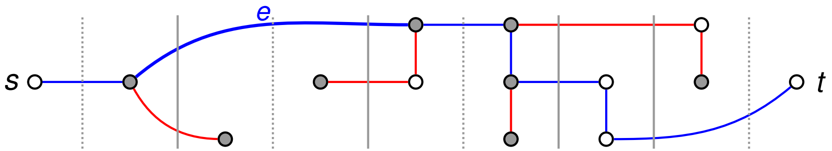

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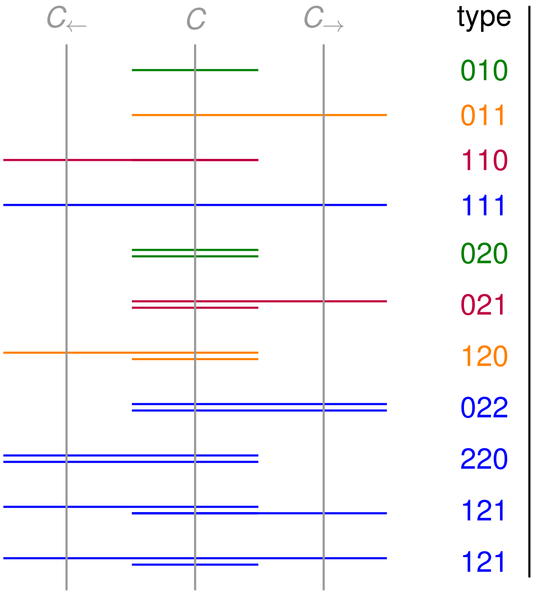
Now:

Distribute according to criticality: C needs $\beta(2-x^*(C))(x^*(C)-1)$

Prevents us from increasing β beyond $\frac{2}{5}$ if, for a cut C, each tree S and its edge $e \in I_S \cap C$, there are critical cuts C', C'' containing e (one which is C) with $|S \cap C'| = 1$ and $|S \cap C''|$ even.

Henceforth: ignore cuts with $x^*(C) \geq 1.73$.

Configurations at a critical cut: edges in $S \cap C$



$C_{\leftarrow}$ and $C_{\rightarrow}$ are the next cuts left and right with $x^*(C) < 1.73$

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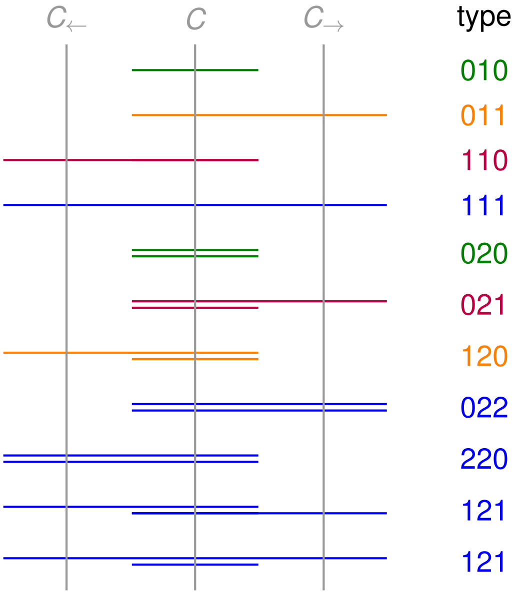

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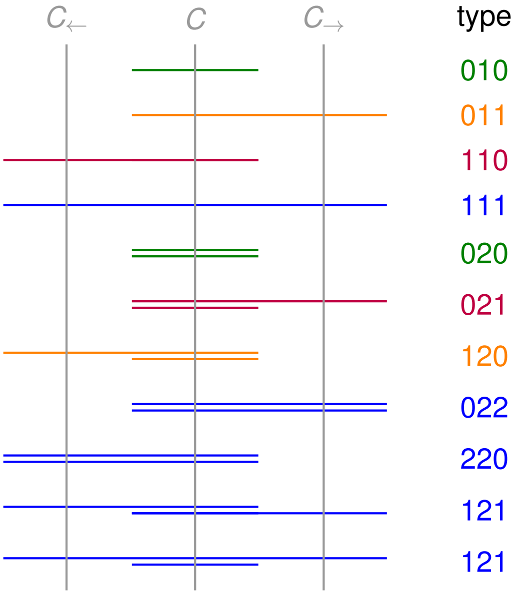

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Note: there are at most 0.73 edges on average in each of $C \cap C_{\leftarrow}$ and $C \cap C_{\rightarrow}$

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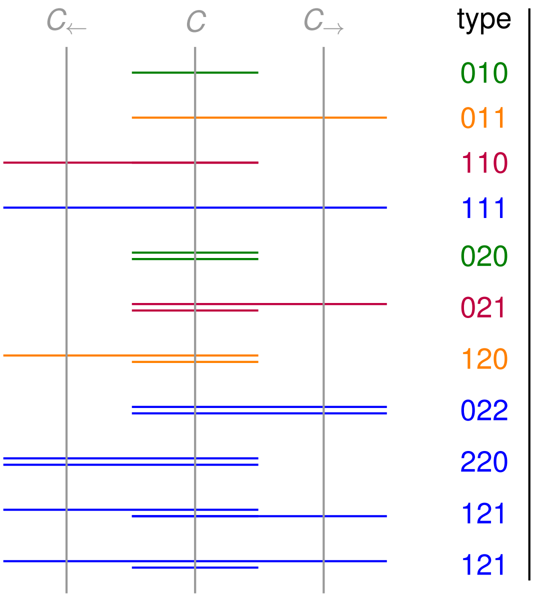

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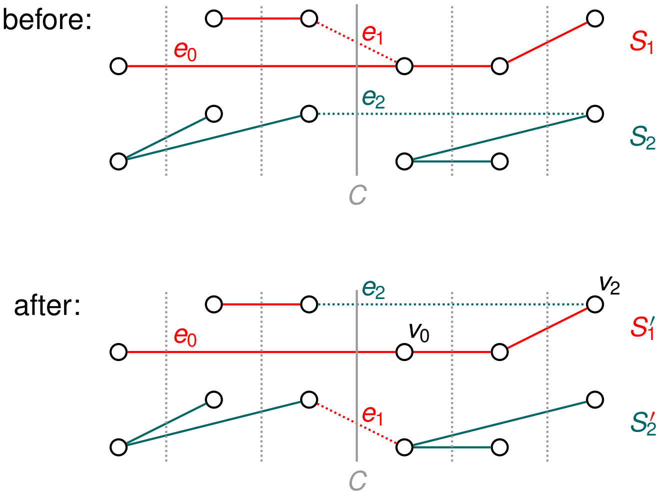
Show:

no 110 or no 021

no 011 or no 120

Reassembling trees: removing a pair 011 and 120



Clean critical cuts off 011/120 from left to right.
Then clean critical cuts off 110/021 from right to left.

Conclusion and open questions

- (My) calculations are rather complicated, also due to less critical cuts, trees with three edges, . . .
- New approximation ratio 1.599
- Same bound on integrality ratio
- Tighter analysis possible, but not close to 1.5.
- Probably need stronger reassembling for much better ratio
- Extension to T-tours for $|T| > 2$ possible?
- Application to other TSP variants?

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Thank you!

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