Bonn Problem-Solving Seminar, BPS 4. August 9, 2013

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(In the solutions, it is feasible to rely on such basic results as Kőnig theorem, Menger theorem, MFMC theorem and algorithm, Dijkstra algorithm, Hoffman's theorem on feasible circulations, etc.)

1. Let $D = (V, A)$ be a planar digraph with a given embedding whose underlying undirected graph is 2-connected. By a face we mean the (not-necessary one-way) circuit surrounding a bounded region of $D$. Prove that the maximum number of edge-disjoint one-way faces of $D$ is equal to the minimum number of edges covering all one-way faces.

2. Let $G_i = (V, E_i)$ be edge-disjoint subgraphs ($i = 1, \ldots, k$) of an undirected graph $G = (V, E)$. Prove that $G$ has a smooth orientation such that its restriction to each $G_i$ is also smooth.

3. Suppose you bought a gadget at the corner-store that is able to compute a cheapest perfect matching in any perfectly matchable bipartite graph admitting a cost function on its edge-set. Show how your gadget can be used for a given integer $k$ to compute a chepaest $k$-element matching of a bipartite graph.

4. Given a weight function on the ground set of a matroid, prove that the bases of maximum weight satisfy the basis axioms. Is it true that tha matroid arising in this way is always graphic provided that the initial matroid is graphic?

5. Suppose that in a digraph $D = (V, A)$ with a specified root node $r_0$, there are $k$ openly disjoint dipaths from $r_0$ to every other node. Then it is possible to colour the edges leaving $r_0$ with $k$ colours in such a way that there are $k$ openly disjont paths from $r_0$ to each node $v \in V - r_0$ so that the $k$ starting edges of the paths are of different colours.