Exercise Set 3

Exercise 3.1. Consider the following local search algorithm for the MAXIMUM CUT problem: Start with an arbitrary vertex set $S \subseteq V$. Iterate the following: If a single vertex can be added to S or can be removed from S such that $|\delta(S)|$ increases, do so. If no such vertex exists, terminate and return $\delta(S)$.

(a) Prove that this algorithm is a 2-approximation algorithm. (In particular, show that it runs in polynomial time.)

(b) Does the algorithm always find an optimum solution for planar graphs or bipartite graphs?

(c) Give a linear time 2-approximation algorithm for the MAXIMUM CUT problem in graphs with nonnegative edge weights.

(6 points)

Exercise 3.2. Describe exact algorithms with running times of $O(2^{\frac{n}{2}})$ for the following problems:

(a) SUBSET SUM:
Given $K, n, x_1, \ldots, x_n \in \mathbb{N}$, find $S \subseteq \{1, \ldots, n\}$ with $\sum_{i \in S} x_i = K$ (or decide that no such set exists).

(b) KNAPSACK (where n denotes the number of items).

(5 points)

Exercise 3.3. The KNAPSACK PROBLEM can be formulated as integer program:

$$\max \left\{ \sum_{i=1}^{n} c_i x_i \,:\, \sum_{i=1}^{n} w_i x_i \leq W,\ x_i \in \{0,1\}\ \forall\, 1 \leq i \leq n \right\} \tag{1}$$

For an instance $\mathcal{I}$, denote the optimum of (1) by $\mathrm{OPT}(\mathcal{I})$ and let $\mathrm{LP}(\mathcal{I})$ be the optimum of the linear relaxation, where $x_i \in \{0,1\}$ is replaced by $0 \leq x_i \leq 1$.

Show that the *integrality gap*

$$\sup_{\mathcal{I}} \left\{ \frac{\mathrm{LP}(\mathcal{I})}{\mathrm{OPT}(\mathcal{I})} \,:\, \mathrm{OPT}(\mathcal{I}) \neq 0 \right\}$$

of the Knapsack Problem is unbounded. What is the integrality gap of the Knapsack Problem restricted to instances with $w_i \le W$ for all $i = 1, \dots, n$?

(3 points)

Exercise 3.4. (a) Consider the Fractional Multi Knapsack Problem: Given natural numbers $n, m \in \mathbb{N}$ and w_i, $c_{ij} \in \mathbb{N}$ as well as $W_j \in \mathbb{N}$ for $1 \le i \le n$ and $1 \le j \le m$, find x_{ij} satisfying $\sum_{j=1}^{m} x_{ij} = 1$ for all $1 \le i \le n$ and $\sum_{i=1}^{n} x_{ij} w_i \le W_j$ for all $1 \le j \le m$ such that $\sum_{i=1}^{n} \sum_{j=1}^{m} x_{ij} c_{ij}$ is minimum.

State a polynomial-time combinatorial algorithm for this problem.
(Do not use that a linear program can be solved in polynomial time.)

(b) Can we solve the integral Multi Knapsack Problem in pseudopolynomial time if m is fixed?

(6 points)

Deadline: Thursday, May 3rd, before the lecture. The websites for lecture and exercises can be found at:

`http://www.or.uni-bonn.de/lectures/ss18/appr_ss18_ex.html`

In case of any questions feel free to contact me at traub@or.uni-bonn.de.