Linear and Integer Optimization
Assignment Sheet 4
Inofficial English Translation

1. For a polytope $P = \{x \in \mathbb{R}^n \mid Ax \leq b\} \neq \emptyset$ let $P' := \{(x,t) \in \mathbb{R}^n \times \mathbb{R} \mid Ax \leq tb, 0 \leq t \leq 1\}$.

   (a) Show that $P' = \mathrm{conv}((P \times \{1\}) \cup \{0\})$.

   (b) Prove that for each face F of P the set $\mathrm{conv}((F \times \{1\}) \cup \{0\})$ is a face of P'.

   (c) Do these these statements still necessarily hold if P is an unbounded polyhedron? (2+2+1 points)

2. For $n \in \mathbb{N} \setminus \{0\}$ and a subset $X \subseteq \mathbb{R}$ let

$$M_X = \left\{ A = (a_{ij})_{\substack{i=1,\dots,n \\ j=1,\dots,n}} \mid a_{i_0 j_0} \in X, \sum_{i=1}^{n} a_{ij_0} = 1, \sum_{j=1}^{n} a_{i_0 j} = 1 \quad (\text{for } i_0, j_0 \in \{1, \dots, n\}) \right\}.$$

   Show that an $n \times n$-matrix A is in $M_{\mathbb{R}_{\geq 0}}$ if and only if it is a convex combination of matrices in $M_{\{0,1\}}$. (4 points)
   Hint: Induction in n.

3. Let $X \subseteq \mathbb{R}^n$ and $y \in \mathrm{conv}(X)$. Prove that there are vectors $x_1, \dots, x_k \in X$ with $k \leq n+1$ and $y \in \mathrm{conv}(\{x_1, \dots, x_k\})$. (4 points)

4. Let $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ be a polyhedron. Moreover, let $P^* := \{y \in \mathbb{R}^n \mid y^t x \leq 1 \text{ for all } x \in P\}$ and $P^0 := \{y \in \mathbb{R}^n \mid y^t x \leq 0 \text{ for all } x \in P\}$.

   (a) Show that P^* is a polyhedron.

   (b) Prove that $(P^*)^* = P$ if and only if $b \geq 0$.

   (c) In addition, assume $b = 0$. Show that $P^* = P^0$ and prove that P^0 is the convex cone generated by the rows of A. (2+3+2 points)

Due date: Thursday, May 5, 2022, before the lecture in the lecture hall.