Research Institute for Discrete Mathematics
University of Bonn

Linear and Integer Optimization
Assignment Sheet 3

1. Let $X \subseteq \mathbb{R}^n$ and $y \in \text{conv}(X)$. Prove that there are vectors $x_1, \ldots, x_k \in X$ with $k \le n+1$ and $y \in \text{conv}(\{x_1, \ldots, x_k\})$. (5 points)

2. Let P be a polyhedron with $\dim(P) = d$ and F a face of P with $\dim(F) = k \in \{0, \ldots, d-1\}$. Show that there are faces $F_{k+1}, F_{k+2}, \ldots, F_{d-1}$ of P with
   i) $F \subseteq F_{k+1} \subseteq F_{k+2} \subseteq \cdots \subseteq F_{d-1} \subseteq P$,
   ii) $\dim(F_{k+i}) = k + i$ for $i \in \{1, \ldots, d-k-1\}$. (5 points)

3. For $n \in \mathbb{N} \setminus \{0\}$ and a subset $X \subseteq \mathbb{R}$ let
$$M_X = \left\{ A = (a_{ij})_{\substack{i=1,\ldots,n \\ j=1,\ldots,n}} \mid a_{i_0 j_0} \in X, \sum_{i=1}^{n} a_{i j_0} = 1, \sum_{j=1}^{n} a_{i_0 j} = 1 \quad (\text{for } i_0, j_0 \in \{1, \ldots, n\}) \right\}.$$
   Show that an $n \times n$-matrix A is in $M_{\mathbb{R}_{\ge 0}}$ if and only if it is a convex combination of matrices in $M_{\{0,1\}}$. (6 points)

4. In the job assignment problem, n jobs with execution times $t_1, ..., t_n \in \mathbb{R}_{\ge 0}$ need to be processed by m workers. For each job i we are given by $S_i \subseteq \{1, ..., m\}$ the set of workers that are qualified to perform job i. It is possible for several workers to process the same job in parallel to speed up the process but one worker can only process one job at a time.
   (a) Formulate an LP minimizing the *makespan* for processing all jobs (the time until the last worker finishes).
   (b) Dualize this LP.
   (c) Develop a simple polynomial time algorithm for $n = 2$ that finds an optimal solution (for the primal problem) and prove its correctness. (3+3+3 points)

Due date: Tuesday, November 17, 2015, before the lecture.