Research Institute for Discrete Mathematics
University of Bonn

Linear and Integer Optimization
Assignment Sheet 7

1. Define $\|A\| := \max_{\|x\|=1} \|Ax\|$ for $A \in \mathbb{R}^{n\times n}$, where $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$ is the standard Euclidean norm. Prove:

   (a) $\|A\|$ is a norm

   (b) $\|aa^T\| = a^T a$

   (c) $\|A\| = \max\{x^T Ax \mid \|x\| = 1\}$ if A is positive semidefinite

   (d) $\|A\| \leq \|A + B\|$ if A and B are positiv semidefinite (2+2+2+2 points)

2. Show that $|\det(A)| \leq \prod_{i=1}^n \|a_i\|$ for an $n \times n$-matrix A with columns $a_1, \dots, a_n$ (where $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$ is again the standard Euclidean norm). (3 points)

3. A semidefinite program is an optimization problem

$$\begin{aligned} \min\ & C \star X & \\ A_i \star X &\leq b_i & \forall i = 1, \dots, m \\ X &\succeq 0 & \\ X &\in \mathbb{R}^{n\times n} & \end{aligned}$$

   where $C, A_1, \dots, A_n$ are matrices, $A \star X := \sum_{1\leq i,j\leq n} a_{ij}x_{ij}$ and $X \succeq 0$ means that X is symmetric and positiv semidefinite.

   (a) Show that the set $\{X \in \mathbb{R}^{n\times n} \mid X \succeq 0\}$ is a closed cone.

   (b) Construct a polynomial-time separation oracle for this set. (You may assume that you can compute basic arithmetic operations on real numbers, including square roots, exactly and in constant time.) (4+4 points)

4. Consider the following optimization problem:

$$\begin{aligned} \min\ & \frac{(c^t x)^2}{d^t x} \\ \text{s.t.}\quad Ax &\geq b \\ x &\geq 0 \end{aligned}$$

   where $c, d \in \mathbb{R}^n$, $A \in R^{m\times n}$ and $b \in \mathbb{R}^m$ are given such that $d^t x > 0$ for any $x \in \mathbb{R}^n$ with $Ax \geq b$ and $x \geq 0$. Show that this problem can be written as a semidefinite program (see the previous exercise). (6 points)
   Hint: Replace the objective function by "k" (where k is a new variable) and add the constraint $k \geq \frac{(c^t x)^2}{d^t x}$.

Due date: Tuesday, December 15, 2015, before the lecture.