Linear and Integer Optimization
Assignment Sheet 12

1. (a) Show that a non-empty polyhedral cone C is pointed if and only if there is a vector b such that $b^t x > 0$ for all $x \in C \setminus \{0\}$.

   (b) Let C b a non-empty rational pointed polyhedral cone. Show that there is a unique minimum integral Hilbert basis generating C. (4+5 points)

   Hint: For (b) consider the integral vectors in C that cannot be written as the sum of two other integral vectors in C.

2. Show that each unimodular square matrix arises from the identity matrix by a series of elementary unimodular column operations. (4 points)

3. Show that $A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ is not totally unimodular but $\{x \in \mathbb{R}^3 \mid Ax = b\}$ is integral for all integral vectors b. (5 points)

4. Consider the following capacitated facility location problem: given a set of clients C and a set of potential facility locations F, a metric ℓ on $C \cup F$ representing connection costs, facility opening costs $p : F \to \mathbb{R}_{\geq 0}$ and capacities $c : F \to \mathbb{N}$, and client demands $d : C \to \mathbb{N}$, find a set $I \subseteq F$ of facilities to be opened and an assignment $f : C \to I$ of clients to open facilities such that the capacity bounds are respected ($\sum_{c \in f^{-1}(x)} d(c) \leq c(x)$) and the sum of opening costs and connection costs of clients to their assigned facilities is minimized.

   (a) Model this problem as an integer program.

   (b) Give a non-trivial example instance for which the LP relaxation of your IP has a unique optimum solution which is integral. Give an example instance for which every optimal solution of the LP relaxation is fractional. (3+4 points)

Due date: Tuesday, February 2, 2016, before the lecture.