Research Institute for Discrete Mathematics
University of Bonn

Linear and Integer Optimization
Assignment Sheet 13

1. Let $A \in \{0,1\}^{m\times n}$ be a matrix where in each column the 1's are arranged consecutively, i.e. for each column $j \in \{1,\dots,n\}$ there are $i_1^j, i_2^j \in \{1,\dots,m\}$ s.t.:

$$a_{ij} = \begin{cases} 1, & i_1^j \le i \le i_2^j \\ 0, & \text{else} \end{cases}$$

for $j \in \{1,\dots,n\}$ and $i \in \{1,\dots,m\}$ (if $i_1^j > i_2^j$, the column consists of zeros only). Show that A is totally unimodular.

2. Consider the following problem: We are given a directed graph G and nodes $s,t \in V(G)$ with $s \neq t$. Moreover, we are given integral mappings $l,u : E(G) \to \mathbb{Z}$ such that $l(e) \le u(e)$ for all $e \in E(G)$. The task is to find a mapping $f : E(G) \to \mathbb{R}$ with $l(e) \le f(e) \le u(e)$ for all edges $e \in E(G)$ and $\sum_{e\in\delta_G^-(v)} f(e) = \sum_{e\in\delta_G^+(v)} f(e)$ for all $v \in V(G)\setminus\{s,t\}$ such that $\sum_{e\in\delta_G^+(s)} f(e) - \sum_{e\in\delta_G^-(s)} f(e)$ is maximimized. This problem generalizes the max-flow problem. Show that there is always an integral optimum solution and show that the value of a maximum solution equals

$$\min\left\{ \sum_{e\in\delta_G^+(X)} u(e) - \sum_{e\in\delta_G^-(X)} l(e) \mid X \subseteq V(G)\setminus\{t\}, s \in X \right\}.$$

3. Use the previous exercise to prove Dilworth's Theorem: in every partially ordered set $(X,\le)$, the maximum size of an antichain (= set of pairwise incomparabe elements) equals the minimum number of chains (= sets of pairwise comparable elements) that are needed to cover X.

4. (a) Give an example of a polyhedron with $P_I \neq P^{(i)}$ for all $i \in \mathbb{N}$.
   (b) Show that for any $k \in \mathbb{N}$ there is a rational polyhedron such that $P_I \neq P^{(i)}$ for all $i \in \{1,\dots,k\}$.